Information Systems Security

Lecture 4

Asymmetric cryptography

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references

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Outline

1. Basic mathematical concepts
2. Public key cryptography
3. OWF
4. RSA
5. ElGamal

1. The modulo operation

- Definition
  - Let a, r, n be integers and let $q > 0$
  - We write $a \equiv r \bmod n$ if n divides $a - r$ (or $r - a$) and $0 \leq r < n$
  - n is called the ***modulus***
  - r is called the ***remainder***
    - Note that r is positive or zero
  - Note that $a = n.q + r$ where q is another integer (***quotient***)

- Example: $42 \equiv 6 \bmod 9$
  - 9 divides 42 - 6 = 36
  - 9 also divides 6 - 42 = -36
  - Note that 42= 9x4 + 6
    - ($q = 4$)

Number Theory

- Natural numbers $N = \{1,2,3,\ldots\}$
- Whole numbers $W = \{0,1,2,3, \ldots\}$
- Integers $Z = \{\ldots,-2,-1,0,1,2,3, \ldots\}$

- Divisors
  - A number b is said to divide a if $a = mb$ for some m where $a,b,m \in Z$
  - We write this as $b|a$
    - Read as "b divides a"

Divisors

- Some common properties
  - If $a|1$, $a = +1$ or -1
  - If $a|b$ and $b|a$ then $a = +b$ or $-b$
  - Any $b \in Z$ divides 0 if $b \neq 0$
  - If $b|g$ and $b|h$ then $b|(mg + nh)$ where $b,m,n,g,h \in$ Z

- Examples:
  - The positive divisors of 42 are 1,2,3,6,7,14,21,42
  - $3|6$ and $3|21 \Rightarrow 3|21m+6n$ for $m,n \in Z$

Prime Numbers

- An integer *p* is said to be a prime number if its only positive divisors are 1 and itself
  - Examples 2, 3, 7, 11, ..
- Any integer can be expressed as a ***unique*** product of prime numbers raised to positive integral powers
  - $n=p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}$ // n: ingterger, p_i:prime, e_i: positive integer
- Examples
  - $7569 = 3 \times 3 \times 29 \times 29 = 3^2 \times 29^2$
  - $5886 = 2 \times 27 \times 109 = 2 \times 3^3 \times 109$
- This process is called ***Prime Factorization***

Greatest common divisor (GCD)

- Definition: ***Greatest Common Divisor***
  - This is the largest divisor of *both a* and *b*
- Given two integers *a* and *b*, the positive integer *c* is called their GCD or greatest common divisor if and only if
  - $c \mid a$ and $c \mid b$
  - Any divisor of both *a* and *b* also divides *c*
- Notation: *gcd(a, b) = c*
- Example: *gcd(49,63)* = ?

- *gcd(a,b)=gcd(b, a mod b)*
- Exception: *gcd(0,0)=0*

Relatively Prime Numbers

- Two numbers are said to be ***relatively prime*** if their *gcd* is 1
  - Example: 63 and 22 are relatively prime

- How do you determine if two numbers are relatively prime?
  - Find their *gcd* or
  - Find their prime factors
    - If they do not have a common prime factor other than 1, they are relatively prime

  - Example: $63 = 9 \times 7 = 3^2 \times 7$ and $22 = 11 \times 2$

Modular Arithmetic Again

- We say that $a \equiv b \bmod m$ if $m \mid a - b$
  - Read as: a is ***congruent*** to b modulo m
  - m is called the *modulus*
  - Example: $27 \equiv 2 \bmod 5$
- Note that b is the ***remainder*** after dividing a by m
  - Example: $27 \equiv 2 \bmod 5$ and $7 \equiv 2 \bmod 5$
- $a \equiv b \bmod m \Rightarrow b \equiv a \bmod m$
  - Example: $2 \equiv 27 \bmod 5$
- We usually consider the *smallest positive remainder* which is sometimes called the ***residue***

Modulo Operation

- The modulo operation "**reduces**" the infinite set of integers to a finite set

- Example: modulo 5 operation
  - We have five sets
    - {…,-10, -5, 0, 5, 10, …} => $a \equiv 0 \bmod 5$
    - {…,-9, -4, 1, 6, 11,…} => $a \equiv 1 \bmod 5$
    - {…,-8, -3, 2, 7, 12,…} => $a \equiv 2 \bmod 5$
    - {…,-7, -2, 3, 8, 13,…} => $a \equiv 3 \bmod 5$
    - {…,-6, -1, 4, 9, 14…} => $a \equiv 4 \bmod 5$
  - The set of residues of integers modulo 5 has five elements {0,1,2,3,4} and is denoted Z_5.

Euler phi (or totient) function

- For $n \geq 1$, $\phi(n)$: is the number of integers in [$1,n$] which are relatively prime to n // $\phi(n)$ is the ***Euler phi*** or ***totient function***

- If p is prime, then $\phi(p)=p-1$

- If $gcd(m,n)=1$, then $\phi(mn)= \phi(m).\phi(n)$

- Examples:
  - $\phi(21)= \phi(3).\phi(7) = (3-1) * (7-1) =12$

multiplicative group $\mathbf{Z_n}^*$

- Definition: the ***multiplicative group*** *Z_n^* of Z_n*
  - $Z_n^*=\{a\in Z_n \mid gcd(a,n)=1\}$
  - If n is prime then $Z_n^*=\{a\in Z_n \mid 1\leq a\leq n-1\}$
  - $\phi(n)=|Z_n^*|$

- Let n $\geq$ 2 be an integer
  - Euler's theorem: If $g \in Z_n^*$ then $g^{\phi(n)} \equiv 1 \ (mod\ n)$
  - If n is a product of distinct primes, and if $r \equiv s\ mod\ (\phi(n))$, then $g^r \equiv g^s\ (mod\ n)$ for all integers g
  - *i.e.,* when <u>working modulo an *n*, exponents can be reduced modulo $\phi(n)$</u>

multiplicative group $\mathbf{Z_n}^*$

- Let *p* be a prime nubmer
  - Fermat's theorem: If $gcd(a,p)=1$, then $g^{p-1} \equiv 1 \ (mod\ p)$
  - If $r \equiv s \ mod\ (p-1)$, then $g^r \equiv g^s \ (mod\ p)$ for all integers g
    - *i.e.,* when working modulo a prime *p*, exponents can be reduced modulo *p-1*
  - Particular case: $g^p \equiv g \ (mod\ p)$ for all integers g

Generator of Z_n^*

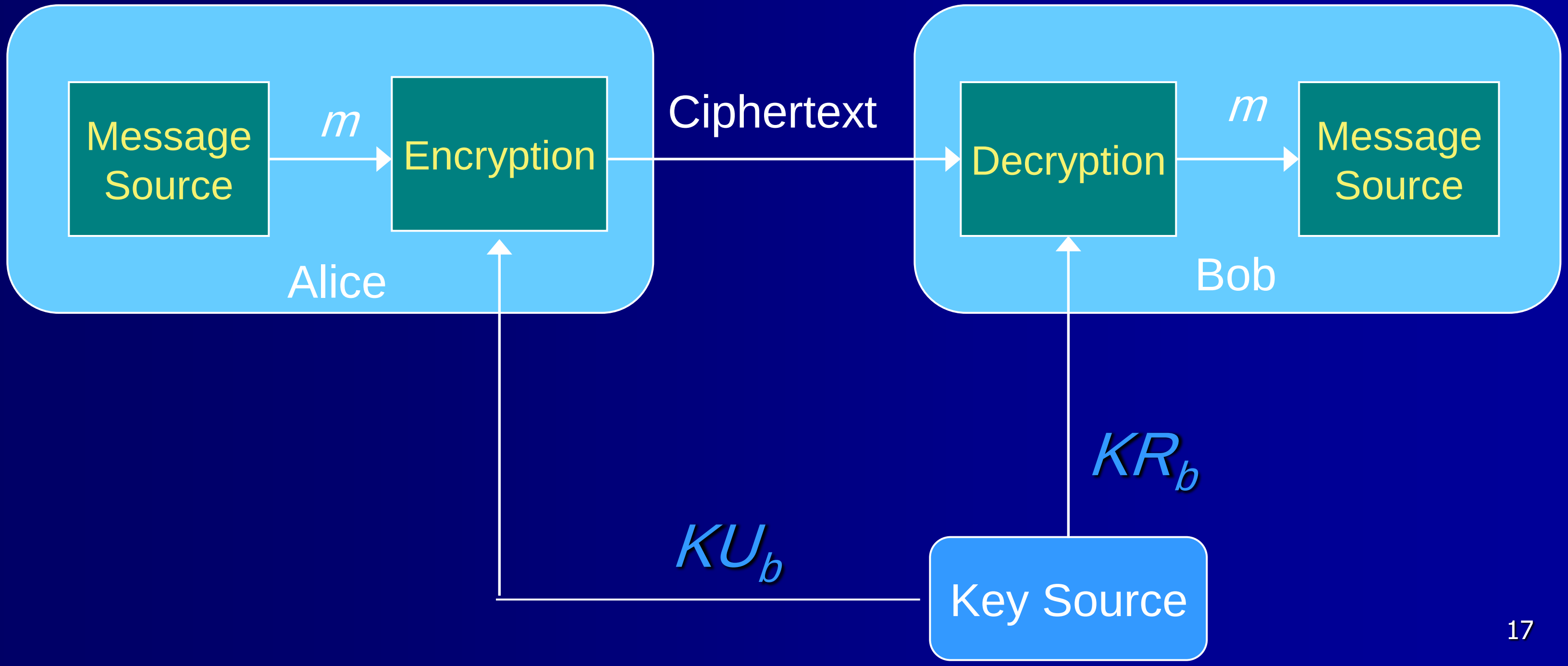
- Let $g \in Z_n^*$, the ***order*** of g is the least positive integer t such that $g^t \equiv 1 \bmod n$
- If the order of $g \in Z_n^*$ is t, and $g^s \equiv 1 \pmod n$, then t divides s
  - A particular case: $t|\phi(n)$
- Let $g \in Z_n^*$, if the order of g is $\phi(n)$, then g is said to be a ***generator*** or a ***primitive element*** of Z_n^*.
  - If g is a generator of Z_n^*, then $Z_n^* = \{g^i \bmod n \mid 0 \le i \le \phi(n) - 1\}$

2. Public-key cryptography

- Called also ***asymmetric cryptography***
- The keys used to encrypt and decrypt are different.
- Anyone who wants to be a receiver needs to "publish" an encryption key, which is known as the ***public key,*** *KU*.
- Anyone who wants to be a receiver needs a unique decryption key, which is known as the ***private key,*** *KR*.
- If B wants to send an enciphered text to A, B should knows ***the encryption algorithm*** and A's public key.

Confidentiality via Public key cryptography

- Alice wants to send a secret message *m* to Bob
- Bob should have 2 keys: public KU_b and private KR_b
- Prior to message encryption, <u>Alice gets by some means an authentic copy of Bob's public key</u> (*i.e.*, the encryption key)



Public-key cryptography

- It should not be possible to deduce the plaintext from knowledge of the ciphertext and the public key.
- It should not be possible to deduce the private key from knowledge of the public key.
- Public-key cryptography is based on ***One-Way Functions***

3. One-Way Functions (OWF)

- A **one-way function** is a function that is "easy" to compute and "difficult" to reverse
- Examples of OWF that we'll use in this lecture to explain public-key systems:
  - Multiplication of two primes
  - Modular exponentiation

OWF: Multiplying two primes

- Multiplication of two prime numbers is **believed** to be a one-way function.
- Given two prime numbers p and q
  - It's easy to find $n=p.q$
  - However, starting from n, it's difficult to find p and q
- Is it prime factorization?

OWF: Modular exponentiation

- The process of ***exponentiation*** just means raising numbers to a power.

- Raising ***a*** to the power ***b***, normally denoted a^b just means multiplying ***a*** by itself ***b*** times. In other words:

$$a^b = a \times a \times a \times \ldots \times a$$

- ***Modular exponentiation*** means computing a^b modulo some other number ***n***. We tend to write this as

$$a^b \bmod n.$$

- Modular exponentiation is “easy”.

OWF: Modular exponentiation

- However, given ***a***, and ***a^b mod n*** (when ***n*** is prime), calculating ***b*** is regarded by mathematicians as a hard problem.

- This difficult problem is often referred to as the **discrete logarithm problem**.

- In other words, given a number ***a*** and a prime number ***n***, the function

$$f(b) = a^b \bmod n$$

is believed to be a one-way function.

4. RSA

- It is named after it inventors Ron Rivest, Adi Shamir and Len Adleman.
- Published in 1978
- It is the most widely used public-key encryption algorithm today.
- It provides confidentiality and digital signatures.
- Its security is based on the difficulty of integer factorization

RSA algorithm (key generation for RSA public-key encryption)

- Each entity A creates a public key and a corresponding private key by doing the following
  - Generate two large (at least 1024 bits) primes p and q
  - Compute $n=pq$ and $\phi(n)=(p-1)(q-1)$.
  - Choose $e < \phi$ relatively prime to ϕ *(i.e., gcd (e, ϕ)=1)*
  - Compute d such that $ed \bmod \phi(n) \equiv 1$
- A's Public key: (e, n) // to be published.
- A's private key: d (*or* (d, n)) // to be kept secretly by A
- Who is capable of computing d?

RSA Encryption/decryption

- Summary: B encrypts a message *m* for A. Upon reception, A decrypts it using its private key.
- Encryption: B should do the following
  - Obtain A's authentic public key *(n,e).*
  - Represent the message as an integer in the interval [*0,n-1*]
  - Compute $c = m^e \bmod n$ // **Encryption**
  - Send the ciphertext c to A
- Decryption: to recover plaintext *m* from *c*, A does the following
  - Use the private key *d* to recover $m = c^d \bmod n$ // **Decryption**
- *How does B obtain A's authentic key?*

Example: confidentiality

- Take $p = 7$, $q = 11$, so $n = 77$ and $\phi(n) = 60$
- Say Bob chooses (KU_b) $e = 17$, making (KR_b) $d = 53$
  - $17 \times 53 \bmod 60 = ?$
- Alice wants to secretly send Bob the message HELLO [07 04 11 11 14]
  - $07^{17} \bmod 77 = 28$
  - $04^{17} \bmod 77 = 16$
  - $11^{17} \bmod 77 = 44$
  - $11^{17} \bmod 77 = 44$
  - $14^{17} \bmod 77 = 42$
- Alice sends ciphertext [28 16 44 44 42]

Example: confidentiality

- Bob receives [28 16 44 44 42]

- Bob uses private key (KR_b), $d = 53$, to decrypt the message:
  - $28^{53} \bmod 77 = 07$ H
  - $16^{53} \bmod 77 = 04$ E
  - $44^{53} \bmod 77 = 11$ L
  - $44^{53} \bmod 77 = 11$ L
  - $42^{53} \bmod 77 = 14$ O

- No one else could read it, as only Bob knows his private key and that is needed for decryption

Attacking RSA

1. Trying to decrypt a ciphertext without knowledge of the private key
   - The encryption process in RSA involves computing the function $c = m^e \bmod n$, which is regarded as being easy
   - An attacker who observes this ciphertext c, and has knowledge of e and n, needs to try to work out what m is.
   - *i.e.*, find m such that $\boldsymbol{m^e} = c \bmod n$
   - In other words, find the e^{th} root of c mod n

- Computing m from c, e and n is regarded as a hard problem and known as ***RSA problem***.

Attacking RSA

2. If the attacker knows the public key of a user (e,n), what would she/he need to do in order to obtain the corresponding private key?
   - He/she needs to find *d* such that $ed \bmod \phi(n) = 1$
   - *i.e.*, needs to know *p* and *q*
   - In other words, he/she must factor *n* (problem of prime factorization)

- Recommended size of n:
  - 768-bit is recommended
  - 1024-bit or larger is required for long term security
  - it is believed that factoring a 512 bit number is about as hard as searching for a 56 bit symmetric key.

5. El Gamal

- ElGamal is another public-key encryption

- We will also take a look at the ElGamal public key cipher system for a number of reasons:
  - To show that RSA is not the only public key system
  - To exhibit a public key system based on a different one way function
  - ElGamal is the basis for several well-known cryptosystems

ElGamal algorithm (key generation)

- Key generation for ElGamal public-key encryption
- Each entity *A* creates a public key and a corresponding private key.
  - Generate a large prime number p (1024 bits)
  - Generate a generator g of the multiplicative group Z_p^* of the integers modulo p
  - Select a random integer x, $1 \le x \le p-2$
  - Compute $y = g^x \bmod p$
  - A's public key is (p, g, y)
    - To be published
  - A's private key is x
    - To be kept secret by A

ElGamal algorithm (key generation)

- Example
- Step 1: Let $p = 2357$
- Step 2: Select a generator $g = \mathbf{2}$ **of** $\boldsymbol{Z_{2357}^{*}}$
- Step 3: Choose a private key $x = \mathbf{1751}$
- Step 4: Compute $y = 2^{\mathbf{1751}} \pmod{\mathbf{2357}}$

  $= \mathbf{1185}$

- Public key is **(2357,2,1185)**
- Private key is **1751**

ElGamal algorithm (Encryption/decryption)

- Summary: B encrypts a message *m* for A, which A decrypts
- Encryption: B should de the following
  - Obtain A's authentic public key *(p, g, y).*
  - Represent the message as an integer in the interval [$0,p-1$]
  - Select an integer *k*, $1 \le k \le p-2$
  - Compute $\gamma = g^k \bmod p$ and $\delta = m.(y)^k \bmod p$
  - *Send the ciphertext* $c = (\gamma, \delta)$ *to* A
- *Decryption*
  - A uses the private key *x* to compute $z = \gamma^{p-1-x} \bmod p$
  - A computes $z.\delta \bmod p\ (=m)$

ElGamal algorithm (Encryption/decryption)

- Encryption
  - To encrypt $m = 2035$ using Public key **(2357,2,1185)**
  - Generate a random number k = **1520**
  - Compute $\gamma = \mathbf{2}^{1520} \bmod \mathbf{2357} = \mathbf{1430}$

    $\delta = 2035 \text{ x } \mathbf{1185}^{\mathbf{1520}} \bmod \mathbf{2357} = \mathbf{697}$

  - Ciphertext $c = (\mathbf{1430}, \mathbf{697})$

- Decryption
  - $z = \gamma^{p-1-x} \bmod p = 1430^{605} \bmod 2357 = 872$
  - $872 \text{x} 697 \bmod 2357 = 2035$

ElGamal Properties

- There is a *message expansion* by a factor of 2
  - *i.e.,* the ciphertext is twice as long as the corresponding plaintext
- *Requires a random number generator (k)*
- Relies on discrete algorithm problem, *i.e.*, having $y = g^x \bmod p$ it's hard to find x (the private key)
- ElGamal encryption is randomized (coming from the random number *k*), RSA encryption is deterministic.
- ElGamal is the basis of many other algorithms (*e.g.*, DSA)

Summary

- RSA is a public key encryption algorithm whose security is believed to be based on the problem of factoring large numbers.

- ElGamal is a public key encryption algorithm whose security is believed to be based on the discrete logarithm problem.

- RSA is generally favoured over ElGamal for practical rather than security reasons.

- RSA and ElGamal are <u>less efficient and fast</u> to operate than most symmetric encryption algorithms because they involve modular exponentiation.
  - Public key cryptography confined to **key management** and **signature applications**.

Questions?